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The Stability of Coupled-Core Nuclear Reactors

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SUMMARY

A Liapunov functional is proposed for the stability analysis of coupled-core nuclear reactors. Subsequently a set of stability criteria and stability regions in function space are determined. The method is more powerful than those relying on the use of ordinary Liapunov functions.

1. Introduction

Coupled-core reactors consist of several independently subcritical cores. The neutronic coupling between them makes the entire system critical. Due to the time delays caused by neutron transport between cores, the reactor is described by a differential equation with retarded arguments of the form

$$\dot{u}(t) = F[u(t), u(t-T_1), \ldots, u(t-T_k)] \tag{1}$$

where the T_i are positive constants. If r is the maximum of all T_i, $i=1, \ldots, k$, then the solutions of (1) can be considered as trajectories in $C=C([-r, 0], E^n)$ which denotes the space of continuous functions with domain $[-r, 0]$ and range in E^n, the real Euclidean space of n-vectors. If $u(t, \varphi)$ is a solution of (1) with initial condition φ in C at $t=0$, then the state at time t is denoted by

$$u_t(\theta) = u(t+\theta, \varphi), \qquad -r \leqq \theta \leqq 0, \quad \text{and} \quad u_t(\theta) \in C.$$

A proper extension of concepts in Liapunov theory leads to the construction of regions of stability in C for eqn. (1). A detailed discussion of the application of Liapunov functionals to equations of the considered type is given by Hale [1].

2. The Selection of a Liapunov Functional

A point reactor model with one group of delayed neutrons in each one of N cores yields the following set of reactor equations (see Weaver, [4]).

$$\dot{n}_i(t) = \frac{\rho_i}{\Lambda} n_i(t) - \frac{\beta}{\Lambda} n_i(t) + \lambda C_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\alpha_{ij}}{\Lambda} n_j(t - T_{ij}) \tag{2.1}$$

$$\dot{C}_i(t) = \frac{\beta}{\Lambda} n_i(t) - \lambda C_i(t) \tag{2.2}$$

$$\dot{T}_i(t) = \frac{1}{K_i} n_i(t) - a_i T_i(t) \qquad i=1, \ldots, N \tag{2.3}$$

where $\alpha_{ij} = \alpha_{ji} \geqq 0$ is the coupling coefficient between the ith and the jth core and $T_{ij} = T_{ji}$ the delay time associated with this coupling. The reactivity ρ_i in the ith core is assumed to depend on the neutron density n_i and a temperature T_i, the latter satisfying a temperature-power relation (2.3). The variables that translate the system operating point to zero are

$$x_i = (n_i - n_{i0})/n_{i0}\ ; \quad y_i = (C_i - C_{i0})/C_{i0}\ ; \quad z_i = (T_i - T_{i0})/T_{i0}$$

$$\rho_i = \rho_{i0} + \delta_i(x_i, z_i)\,, \qquad \delta_i(0,0) = 0$$

where n_{i0}, C_{i0}, T_{i0} and ρ_{i0} are the power level, delayed neutron precursor density, average core temperature and reactivity of the ith core at equilibrium. Using the shorthand notations

$$x_i = x_i(t)\ ; \quad x_{jT_{ij}} = x_j(t - T_{ij})\ ; \quad \frac{n_{j0}}{n_{i0}}\frac{\alpha_{ij}}{\Lambda} = bk_{cij}\ ; \quad b = \frac{\beta}{\Lambda}$$

and

$$\delta_i(x_i, z_i) = -\beta(k_{pi}x_i + k_{ti}z_i)$$

where $k_{pi} \geqq 0$ and $k_{ti} \geqq 0$ represent reactivity feedback effects directly proportional to neutron density and to temperature, the kinetic equations (2) become

$$\dot{x}_i = -b(k_{pi}x_i + k_{ti}z_i)(1 + x_i) - bx_i + by_i - b\sum_{\substack{j=1\\ j\neq i}}^{N} k_{cij}x_i + b\sum_{\substack{j=1\\ j\neq i}}^{N} k_{cij}x_{jT_{ij}} \tag{3.1}$$

$$\dot{y}_i = \lambda(x_i - y_i) \tag{3.2}$$

$$\dot{z}_i = a_i(x_i - z_i) \qquad i = 1, \ldots, N\,. \tag{3.3}$$

Let V be a continuous scalar function on C, of the form

$$V = \sum_{i=1}^{N} (K_{1i}x_i^2 + K_{2i}y_i^2 + K_{3i}z_i^2) + \sum_{\substack{i,j=1\\ i\neq j}}^{N} v_{ij}\int_{-T_{ij}}^{0} x_i^2(t+\theta)\,d\theta\,.$$

Choosing $v_{ij} = bK_{1i}k_{cij}$, $K_{3i}a_i = bK_{1i}k_{ti}$, $\lambda K_{2i} = bK_{1i}$, and selecting K_{1i} such that $K_{1j}k_{cji} = K_{1i}k_{cij}$, which is satisfied for $K_{1i} = n_{i0}^2$, one finds after some manipulations

$$V = \sum_{i=1}^{N} n_{i0}^2\left(x_i^2 + \frac{b}{\lambda}y_i^2 + \frac{bk_{ti}}{a_i}z_i^2\right) + \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{n_{i0}n_{j0}\alpha_{ij}}{\Lambda}\int_{-T_{ij}}^{0} x_i^2(t+\theta)\,d\theta\,,$$

$$\dot{V} = -b\sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{n_{i0}n_{j0}\alpha_{ij}}{\Lambda}(x_i - x_{jT_{ji}})^2 - 2b\sum_{i=1}^{N} n_{i0}^2x_i^2(k_{pi}x_i + k_{ti}z_i + k_{pi})$$

$$-2b\sum_{i=1}^{N} k_{ti}n_{i0}^2z_i^2 - 2b\sum_{i=1}^{N} n_{i0}^2(x_i - y_i)^2\,,$$

where the derivative has been taken along the solutions of (3). V will be used as a Liapunov functional for eqn. (3).

3. Reactivity Feedback Proportional to Neutron Density

In this case $k_{ti} = 0$ and $\dot{V}$ is negative semidefinite, since $x_i \geqq -1$. For $\varphi \in C$ and $|\varepsilon|$ sufficiently small we have

$$V(\varphi) \geqq \varepsilon^2|\varphi(0)|^2 = \varepsilon^2\sum_{i=1}^{N}(x_i^2 + y_i^2 + z_i^2)\,.$$

Furthermore one readily sees that the largest invariant set where $\dot{V} = 0$, consists of the union of the full power equilibrium state ($x_i = y_i = z_i = 0$) and the null power equilibrium state ($x_i = y_i = z_i = -1$). Hence using a result by Hale [1] we have

Theorem 1. *All trajectories approach either the equilibrium state at full power or the equilibrium state at zero power as $t \to +\infty$.*

This result implies that sustained oscillations in the reactor system are impossible. Next let us try to estimate the region of attraction of the operating point. In the appendix a sufficient condition is found in order that no trajectory approaches the zero power equilibrium state, yielding the following

Theorem 2. *If for at least one value of* i $(1 \leqq i \leqq N)$

$$k_{pi} > \sum_{\substack{j=1 \\ j \neq i}}^{N} k_{cij} \tag{4}$$

then all trajectories, except the zero power equilibrium state, approach the full power equilibrium state as $t \to +\infty$.

If no value of i can be found such that (4) is satisfied, a region of attraction of the operating point can be constructed by estimating the value of V at null power. One has

$$V_{\text{null power}} = R_0 = \sum_{i=1}^{N} n_{i0}^2 \left(1 + \frac{b}{\lambda}\right) + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{n_{i0} n_{j0} \alpha_{ij}}{\Lambda} T_{ij} ,$$

from which we find

Theorem 3. *The set* $V < R_0$ *in* C *is a region of attraction of the full power equilibrium state.*

4. Reactivity Feedback Governed by Newton's Law of Cooling

Now it is assumed that $k_{ti} \neq 0$ for at least one i. Consider the set

$$V \leqq R_1 \tag{5}$$

Then *a fortiori*

$$n_{i0}^2 \left(x_i^2 + \frac{bk_{ti}}{a_i} z_i^2\right) \leqq R_1 .$$

If R_{1i} is the minimum of $n_{i0}^2(x_i^2 + bk_{ti} z_i^2/a_i)$ on the set $\Phi_i = k_{pi} x_i + k_{ti} z_i + k_{pi} = 0$, and R_1 is the minimum of R_{1i}, $i = 1, \ldots, N$, then (5) is a region of attraction of the operating point. R_1 can be determined using the method of Lagrange multipliers. One finds

$$R_1 = \min_{i=1,\ldots,N} n_{i0}^2 \frac{k_{pi}^2}{k_{pi}^2 + a_i k_{ti}/b} .$$

So we have

Theorem 4. *The set* $V \leqq R_1$ *is a region of attraction of the full power equilibrium state.*

5. Example

Consider a reactor consisting of two identical cores, operating at the same power level. We have

$$V = n_0^2 \sum_{i=1}^{2} \left(x_i^2 + \frac{b}{\lambda} y_i^2 + \frac{bk_t}{a} z_i^2 + \frac{\alpha}{\Lambda} \int_{-T}^{0} x_i^2(t+\theta) d\theta\right)$$

$$R_1 = n_0^2 k_p^2 \Big/ \left(k_p^2 + \frac{ak_t}{b}\right).$$

The region of stability can be visualized by intersecting it with $\varphi(\theta) = \varphi(0) = \text{constant}$; $y_i = 0$, $i = 1, 2$; $z_2 = x_2 = 0$.

The result is a projection of the stability region on the z_1–x_1 plane.

$$\left(1 + \frac{\alpha T}{\Lambda}\right) x_1^2 + \frac{bk_t}{a} z_1^2 \leqq k_p^2 \Big/ \left(k_p^2 + \frac{ak_t}{b}\right).$$

The projection is shown in fig. 1 for the case $k_p = k_c = k_t = 1$, $b/a = 0.1$ and for $\alpha T/\Lambda = bT = 0$ and 0.2. The result can be improved using Willems' [2] method of open Liapunov surfaces. We shall apply the method to the present example. On the set $\Phi_1 = x_1 + z_1 + 1 = 0$ we find

$$\begin{aligned}\dot{\Phi}_1 &= \dot{x}_1 + \dot{z}_1 = (2a-b)x_1 + a + b + by_1 + bx_{2T} \\ &\geqq (2a-b)x_1 + a - b \,.\end{aligned}$$

Similarly on the set $\Phi_2 = x_2 + z_2 + 1 = 0$ we have $\dot{\Phi}_2 \geqq (2a-b)x_2 + a - b$. Let

$$x_{i0} = -(a-b)/(2a-b) \quad z_{i0} = -(1 + x_{i0}) = -a/(2a-b) \qquad i = 1, 2$$

$$R_2 = n_{i0}^2 \left(x_i^2 + \frac{b}{a} z_i^2 \right) = n_0^2 (a^2 - ab + b^2)/(2a-b)^2 \,.$$

Apparently no trajectory can leave the set $\Psi = (V < R_2, \Phi_1 \geqq 0, \Phi_2 \geqq 0)$. All trajectories starting inside Ψ approach the full power equilibrium state. The projection of Ψ on the z_1–x_1 plane is also shown in fig. 1.

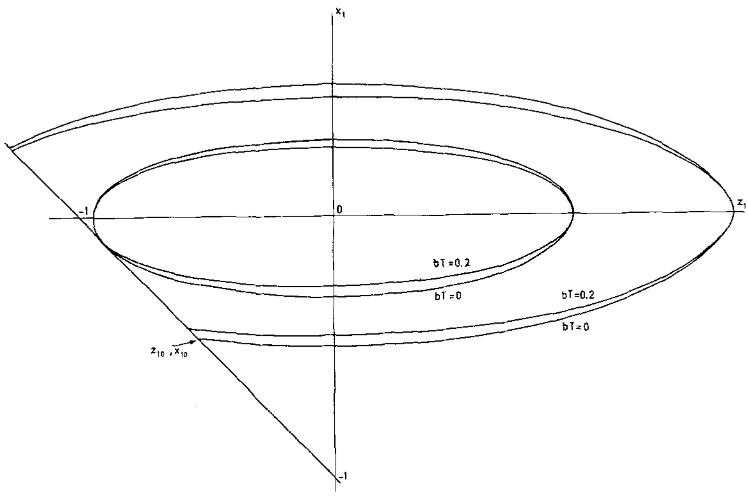

Figure 1. Projection of stability regions on z_1–x_1 plane.

The method can be extended to models having several groups of delayed neutrons and reactivity feedback effects depending on several temperatures in each core.

6. Conclusion

A Liapunov functional has been given for the determination of stability regions for coupled-core reactor systems. These regions are defined as subsets of the space of n-dimensional vector functions over the domain $[-r, 0]$. The results are substantially better than those obtained with ordinary Liapunov functions, as given by Murray [3] and Weaver [4].

7. Appendix

Suppose there exists a solution approaching the null power equilibrium state as $t \to +\infty$. Then for any $\varepsilon > 0$, sufficiently small, and any x_i ($i = 1, \ldots, N$) a time t_0 can be found such that

$$\begin{aligned} x_i(t_0) &= -1 + \varepsilon \\ x_i(t) &< -1 + \varepsilon \quad \text{for all } t > t_0 \,. \end{aligned} \tag{6}$$

Now

$$\begin{aligned} \dot{x}_i + \frac{b}{\lambda}\dot{y}_i &\geqq -b\left(k_{pi}x_i + \sum_{\substack{j=1\\ j\neq i}}^{N} k_{cij}\right)(1+x_i) \\ &\geqq b\left[k_{pi}(1-\varepsilon) - \sum_{\substack{j=1\\ j\neq i}}^{N} k_{cij}\right](1+x_i) \\ &\geqq 0 \qquad t \geqq t_0 \end{aligned}$$

if $|\varepsilon|$ sufficiently small and (4) is satisfied. Hence $x_i+(b/\lambda)y_i$ is non-decreasing for $t\geqq t_0$ and approaches $-1-(b/\lambda)$ as $t\to+\infty$. It follows that

$$x_i + \frac{b}{\lambda}y_i \leqq -1 - \frac{b}{\lambda} \qquad t \geqq t_0$$

Hence

$$\begin{aligned} b(y_i - x_i) &\geqq by_i + b\left(\frac{b}{\lambda}y_i + 1 + \frac{b}{\lambda}\right) \\ &\geqq b\left(1+\frac{b}{\lambda}\right)(y_i+1) \\ &\geqq 0 \end{aligned}$$

and

$$\begin{aligned} \dot{x}_i &\geqq -b\left(k_{pi}x_i + \sum_{\substack{j=1\\ j\neq i}}^{N} k_{cij}\right)(1+x_i) \\ &\geqq b\left[k_{pi}(1-\varepsilon) - \sum_{\substack{j=1\\ j\neq i}}^{N} k_{cij}\right](1+x_i) \\ &\geqq 0 \qquad t \geqq t_0 \end{aligned}$$

which contradicts (6).

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